Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 144 (2016) 1459 – 1468

Procedia Engineering

www.elsevier.com/locate/procedia

12th International Conference on Vibration Problems, ICOVP 2015

Nonlinear Dynamic Analysis of Cracked Cantilever Beam using Reduced Order Model

Kulkarni Atul Shankar[a]*, Manoj Pandey[b]

[a]*M.S. Scholar, Indian Institute of Technology Madras, Chennai 600036, India*
[b]*Assistant Professor, Indian Institute of Technology Madras, Chennai 600036, India*

Abstract

In this work, a reduced order model is obtained for a cracked turbine rotor blade modeled here as a cantilever beam. Accurate dynamical model of this system using a numerical tool such as Finite Element (FE) would typically possess large number of degrees of freedom due to refinement of the mesh near crack and contact, which makes the system computationally intensive especially for long term analysis. We describe a lower order macromodel by using subspace based projection of the full order system to fewer dominant nonlinear normal modes (NNM) of the system, called proper orthogonal modes (POM). Breathing crack is modeled as piecewise linear system with bilinear natural frequency while geometric nonlinearities are incorporated in a cubic Duffing's term. We find that the reduced order model was able to match the original FEM data to the desired accuracy with only first two POD modes of the system and capture the change in frequency introduced by the damage. Two orders of magnitude reduction in the simulation time is obtained. Robustness of the macromodel is checked under different loading conditions viz. changed forcing frequency, pressure loading and damping. Complex nonlinear dynamic effects such as chaos and bifurcations were shown to be captured qualitatively.

Peer-review under responsibility of the organizing committee of ICOVP 2015

Keywords: Nonlinear Dynamics; Fracture; Chaos

* Corresponding author. Tel.: +919028923761;
E-mail address: askulkarni30990@gmail.com

Peer-review under responsibility of the organizing committee of ICOVP 2015
doi:10.1016/j.proeng.2016.06.537

Nomenclature

l	length of the beam, *m*.
b	width of the beam, *m*.
h	height of the beam, *m*.
ρ	density of the beam, kg/m^3.
E	modulus of elasticity, *GPa*.
ζ	damping factor.
v	Poisson's ratio.
a	crack length, *m*.
γ	crack contact parameter.

1. Introduction

Turbo machinery rotors are often operated in harsh environmental conditions such as corrosive atmosphere, high temperature, and high pressure and are subjected to different kinds of loading. These highly stressed conditions result in fatigue, corrosion and creep. These physical phenomena altogether leads to initiation of fatigue crack along with the earlier material irregularities due to heterogeneous nature of material. These fatigue cracks are dangerous such that if not detected in earlier stages, they can cause catastrophic failure of the component. Under subsequent dynamic loading of the cracked structure, the crack opens and closes continuously, leading to the phenomena of breathing crack. This breathing phenomenon induces fatigue stresses in crack front region and leads to propagation of crack.

Dynamical study of such components with presence of breathing crack is complicated due to repetitive opening and closing of crack surfaces leading to a highly nonlinear system. The crack surfaces open and close during each vibration cycle with contact happening between the two, leading to stiffness discontinuity. The other source of nonlinearity is geometric nonlinearity due to large deformation as compared to the geometric dimensions. Accurate analysis of this system, typically with Finite Element methods would generally lead to large number of degrees of freedom especially due to refined mesh near the crack tip and contact behavior between crack surfaces making the problem computationally intensive and costly. Hence, a reduced order scheme is formulated which reduces computational time and cost to a considerable amount.

Here, we study dynamics of a nonlinear cracked cantilevered beam as a simplified model for a cracked rotor blade. Subspace based projection method is used to reduce the modeling space of the system by projecting the dynamics onto the dominant nonlinear normal modes (NNM), hence preserving the nonlinearities in the system. Proper Orthogonal Modes (POM), which are the linear best fit to the NNM and can be obtained from a short but detailed FE analysis of the complete structure and are used as the basis function here. This results in a set of nonlinear ODE's that are solved simultaneously, using numerical means to predict the long term behavior of the system.

Dynamics of cracked structures is an active area of research due to its practical significance. Finite Element based analysis of cracked beam has been performed by H. Nahvi et al [1], Murat Kisa et al [2], while Akira Saito et al [3] and Matthew P. Castanier et al [4], performed the sane for free and harmonically forced conditions of a rotor blade . Simplified models for an open edge cracks in cantilevers have been attempted by R. Rand et al [5], M.H. H. Shen et al [6] and Ugo Andreaus et al [7].

2. Reduced Order Model of Cracked Cantilever Beam

Traditional FEM technique uses local interpolation functions extracted by meshing the domain to obtain approximate solution to the governing PDE. This converts problem from continuous PDE to set of coupled ODEs'. These extracted set of ODEs' possess large number of degrees of freedom and these need to be integrated with respect to time, making it a time consuming method. In case of macromodel approach, global basis functions are extracted from simulation results of FEM simulations. These global basis functions here are obtained using a method called singular value decomposition (SVD) and corresponding macromodel is generated using projection based *Galerkin* method. Finally, set of coupled ODEs' are integrated with respect to time in order to obtain system responses.

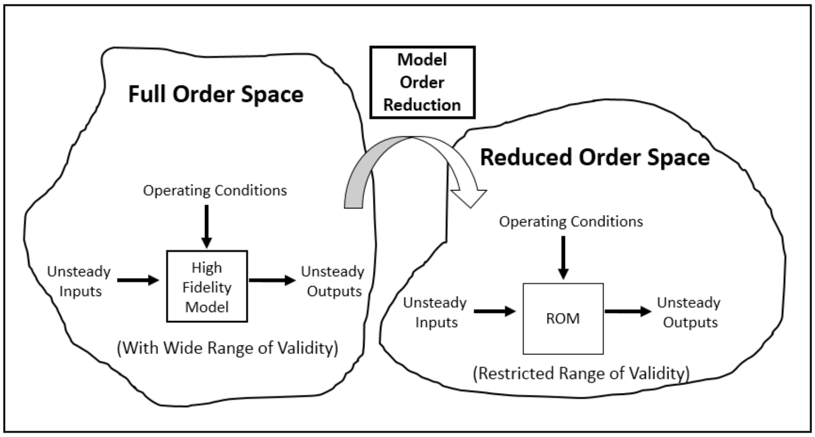


Fig. 1. Reduced Order Model Strategy.

Elmer S Hung et al [10], demonstrated method for obtaining reduced order dynamical models for micromechanical devices using data from simulation results of slow numerical models such as finite element method. A. Chatterjee [11], discussed about the discrete version of the POD, i.e. the singular value decomposition (SVD) of matrices.

2.1. Problem Statement

In this case, a breathing crack is modeled at the upper edge of the cantilever beam as shown in Fig. 2. Dimensions are l = 0.72 m, b = 0.032 m, h = 0.016 m, ρ = 7650 kg/m3, E = 206 GPa, ζ = 0.05, Poisson's ratio = 0.29. A seam crack is modeled with crack length of a = 0.0021 m.

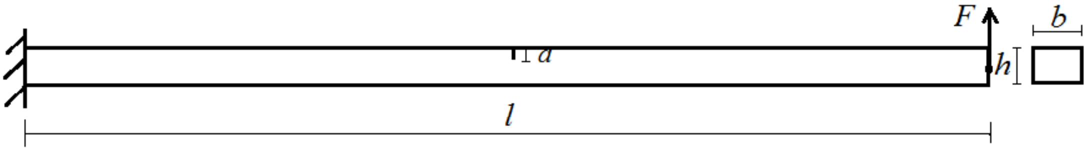


Fig. 2. Cracked Cantilever Beam Model.

The seam crack is modeled in *ABAQUS* by specifying contact properties between the two crack surfaces so that two surfaces do not overlap with each other during each vibration cycle. A periodic load is applied at the free end of the beam.

2.2. Subspace Projection Strategy

A dynamical system can be described with PDE of the form,

$$L(u) = f \tag{1}$$

In Eq. 1, L is any differential operator (linear or nonlinear) and u defines state variables vector i.e. displacement. In the case of Euler Bernoulli beam under large deflection the $L(u)$ is given by [11].

$$EI\left(1-\frac{3}{2}\left(\frac{du}{dx}\right)^2\right)\frac{d^4u}{dx^4}-\frac{3}{2}\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^3u}{dx^3}+C\frac{du}{dx}+\rho A u \tag{2}$$

It is assumed that, state variables vector *u(x, t)* and forcing terms *f(x,t)* are functions of spatial variable *x* and temporal variable *t*. Now, the desired PDE solution *u(x,t)* is approximated in separable form as a series of spatial varying basis functions $a_i(x)$ and temporal coefficients $\alpha_i(t)$ and can be written as,

$$\hat{u}(x,t) = \sum_{i=1}^{N} \alpha_i(t) a_i(x) \tag{3}$$

Where in Eq. 2, LHS is defined as an approximation for *u(x, t)*. N is defined as number of basis functions used. This approximation, results in Eq. 1 to be satisfied only approximately. Now, if orthogonal basis functions are assumed, then by using Galerkin projection method, governing equation of motion for temporal coefficients *αi (t)* are derived. Galerkin method needs the PDE residual $L(u) - f$ to be orthogonal to the global basis functions. This can be stated mathematically as,

$$(a_i, L(u) - f) = \int a_i^T (L(u) - f) dx = 0 \tag{4}$$

The global basis functions for the case of cracked cantilever beam, are obtained from the dynamic content of a short but careful simulation carried out in FE package ABAQUS here. Galerkin method is used to convert governing PDE into set of coupled reduced set of ODE's, expressed in terms of global basis functions. These set of ODE's can further be integrated numerically.

$$M\ddot{\alpha} + C\dot{\alpha} + K\alpha + \beta\alpha^3 + F = 0 \tag{5}$$

Where M, K, C, f are modal mass, stiffness, damping and forcing terms calculated as,

$$M_{ij} = \int_L \rho b_i b_j dx, C_{ij} = \int_L c b_i b_j dx,\ K_{ij} = \int_L (EI \frac{\partial^2 b_i}{\partial x^2} \frac{\partial^2 b_j}{\partial x^2}) dx,$$

$$\beta_{ij} = -\frac{3}{2} \int_L \left[EI\, b_i \left(\left(\frac{\partial b_j}{\partial x} \right)^2 \frac{\partial^2 b_j}{\partial x^2} + 2 \frac{\partial b_j}{\partial x} \frac{\partial^2 b_j}{\partial x^2} \frac{\partial^3 b_j}{\partial x^3} \right) \right] dx,\ F_i = \int_L b_i(F) dx$$

Eq. 5 defines macromodel formulation for a cantilever beam. Modal parameters are calculated and corresponding set of ODEs' are formulated using only first two POD modes. Solution of these set of ODEs' is obtained using numerical integration in *MATLAB*. In this way, macromodel strategy reduces degrees of freedom from thousands to just few global basis function co-ordinates. This will capture system dynamics with lesser POD modes and further improve simulation time and cost.

2.3. Extraction of Global Basis Functions

Nonlinear harmonic response of cracked cantilever beam is obtained from *ABAQUS* with geometrical nonlinearities present in the system. In order to generate displacement basis functions, spatial distribution of state variable *u(x, t)* is sampled at a series of *Ns* different time values during the dynamic simulation. These sampled values of state variable are stored as a set of vectors $\{u_i\}$. The *N* numbers of orthogonal global basis functions $\{b_1, b_2 \ldots b_N\}$ are obtained to represent state variables. These state variables are represented as closely as possible using these selected or extracted orthogonal global basis functions. This can be represented by minimizing the quantity given in Eq. 6 as,

$$\sum_{i=1}^{N_s} \left| u_i - proj(u_i, span\{b_1, \ldots, b_N\}) \right|^2 \tag{6}$$

In Eq. 6, quantity *proj (v,w)* is the projection of *v* on to the subspace *w*. In other sense, the least square measure of error distances between the actually observed state variables and orthogonal global basis functions which represent them is minimized. This can be achieved by just taking singular value decomposition (SVD) of *U* matrix which represents state variables.

$$U = S\sum T^T \tag{7}$$

Where $\Sigma = diag\ (\sigma_1, \ \sigma_2 ... \ \sigma_N)$ is a diagonal matrix, S and T are orthonormal matrices, $\sigma_1 \geqslant \sigma_2 ... \geqslant \sigma_N$, and $N < Ns$. The columns of S are the eigenvectors of UU^T and the columns of T are the eigenvectors of U^TU. SVD approach for generating orthogonal global basis functions is used due to its robustness.

2.4. Formation of Reduced Order Model

M.M.H.Shen [6], discussed modeling of dynamic system with breathing/ fatigue crack present at the top edge of the cantilever beam using a contact parameter. The crack surfaces in case of breathing crack are assumed to be very close and hence there is a contact during vibration. Breathing crack opens when the normal strain at the crack tip is positive and crack will close if the normal strain at the crack tip is negative. When the breathing crack is open, this crack contact parameter $\gamma = 1$ and when it is closed then $\gamma = 0$. The closed crack case can be considered as the uncracked beam case.

$$\gamma = \begin{Bmatrix} 1.........if\ \varepsilon_{xx} > 0 \\ 0.........if\ \varepsilon_{xx} < 0 \end{Bmatrix} \tag{8}$$

From Eq. 8, it is clear that when contact parameter becomes 0 system of equations reduces to Euler-Bernoulli beam equation i.e. no crack is present. When the crack is closed, corresponding natural frequency is ω_i^{uc} and respective mode shape can be written as b_i^{uc} . Whereas, when the crack is open, the corresponding natural frequency is ω_i^c and mode shape is b_i^c . The effective natural frequency of system with both open and closed crack during one cycle is called bilinear natural frequency. The response of this nonlinear dynamic system with breathing crack is obtained as,

$$w_i(x,t) = \gamma b_i^c(x)u_i(t) + (1-\gamma)b_i^{uc}(x)u_i(t) \tag{9}$$

Where b_i^{uc} is i^{th} POD mode shape of uncracked beam and b_i^c corresponds to i^{th} POD mode shape of cracked cantilever beam. Also, u_i represents general coordinate for each mode. This gives the basic formulation of a macromodel for obtaining response of cracked cantilever beam with breathing crack and geometrical nonlinearities. Due to continuous opening and closing of the crack the effective frequency of the beam or bilinear frequency (ω_b) is a combination of the open and closed mode frequencies and is obtained as [5],

$$\omega_b = \frac{2\omega\omega_c}{\omega + \omega_c} \tag{10}$$

Where, ω_b = Bilinear frequency of the beam, ω_c = Natural frequency of the cracked beam, ω = Natural frequency of the uncracked beam.

3. Results and Discussions

First a training signal is generated by dynamic analysis of a representative system, from which the dynamic characteristics are derived. Here forced vibrations of cantilever beam is carried out using harmonic loading as $f = 2000sin\ (596.531t)$. There is no damping in the system. Nonlinear response of the system is obtained at free end at point of loading as shown in Fig. 3. A frequency shift is observed due to breathing crack. This shift is due to decreased stiffness of beam as a result of local flexibility, caused by breathing crack. FFT of the nonlinear response is computed and is compared with FFT of cracked beam response as shown in Fig. 4. Shift in the natural frequency of the system is as expected from the loss of stiffness. Next the macromodel generated for determining nonlinear response of the cracked cantilever beam is presented.

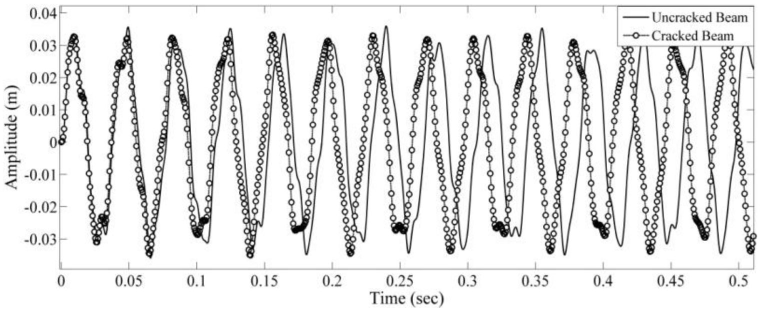


Fig. 3. Comparison of Uncracked and Cracked Responses (Forced Vibrations).

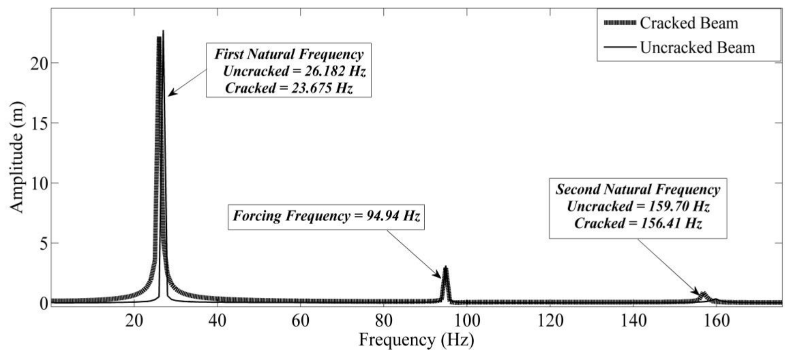


Fig. 4. Comparison of FFT Responses of Uncracked and Cracked Beam.

3.1. Reduced Order Model for Cracked Beam

Orthogonal global basis functions are generated from simulation results of cracked as well as uncracked beam as explained earlier. Snapshot matrix is obtained from response generated using FEM simulation and then SVD of this is taken. *Galerkin* method converts governing PDE of cantilever beam to a set of coupled ODE's which reduces degrees of freedom required to capture the response of the system. Corresponding modal parameters of cracked beam are obtained using Eq. 5. Equations of motion are formulated for first two POD modes. These equations are then numerically integrated using *ODE 45* subroutine in *MATLAB*. For obtaining total response of macromodel with breathing crack, POD modes of both cracked and uncracked beam model are used with extracted orthogonal basis function as given by Eq. 9.

Macromodel response and FEA response shows good agreement with each other as shown in Fig. 5. It is clear that by using only first two POD modes, response of cantilever beam with geometric nonlinearities and breathing crack can be obtained efficiently. An error calculation shows that average error between two plots is of the order of *0.036 %.* While simulation time is reduced by a factor of *215* approximately.

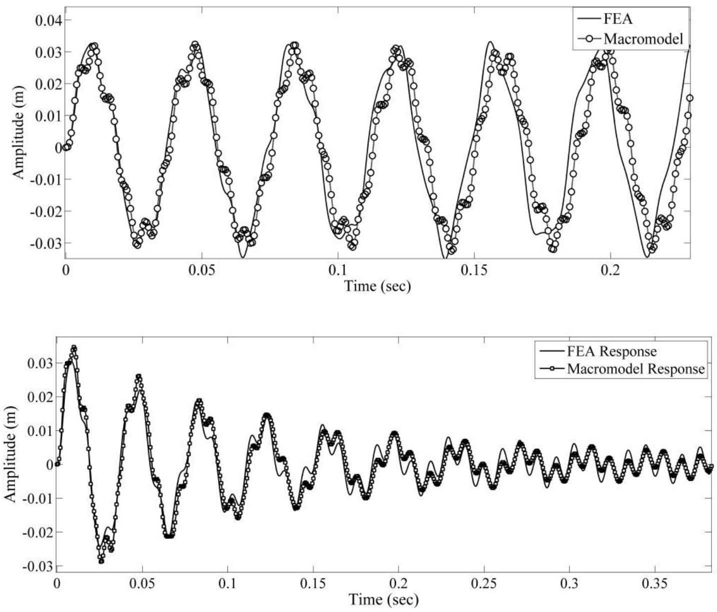


Fig. 5. Comparison of FEA and Macromodel Response of Cracked Beam for a) Undamped b) with Damping ($\zeta = 0.05$).

3.2. Robustness Check of Macromodel

Robustness of this methodology is checked, in order to determine suitability of using same set of basis functions under different loading condition such as damping, changed frequency, changed forcing amplitude and frequency and pressure loading etc. Modal parameters for uncracked and cracked beam problem are given in Table 1. By obtaining mode shapes $b_i(x)$ for first two modes, two ODEs' were formulated as given by Eqs. 5 and 9 and then those two equations were solved numerically in *MATLAB*. Four different loading conditions from that used to determine the training signal are applied to check the robustness of macromodel, keeping other modal parameters same as in initial case and are listed as follows:

Case 1: Macromodel Response of Cracked Beam with Damping ($\zeta = 0.05$). The forcing used is the same as the training signal.

Case 2: Macromodel Response of Cracked Beam with Different Forcing Frequency from the training signal. The forcing considered here is $f = 2000\sin(800t)$.

Case 3: Macromodel Response of Cracked Beam with Different Forcing Amplitude and Frequency. The forcing considered here is $f = 1000\sin(800t)$.

Case 4: Macromodel Response of Cracked Beam with different loading distribution from the training signal. A uniform pressure of the magnitude of *25 kPa* is applied at the upper surface of the beam and response is calculated at the free.

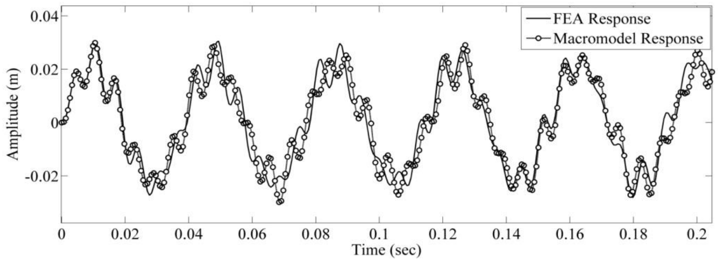
Fig. 6 shows the comparison of, macromodel response of the above cases while Table 2 shows the comparison of simulation time and speed up under of above with good match with the FEA results. The macromodel formulated for breathing crack cantilever beam model is found to be robust enough to be used for different types of the loading conditions. We find that up to *200 %* speed up of the simulations is achieved with good accuracy.

Table 1. Reduced Order Model Parameters for Different Loading Cases of Uncracked Cantilever beam. F1corresponds to an edge load of *2000 N* applied at *95Hz* and *127 Hz* resp. F2 corresponds to a forcing of *1000N* applied at a frequency of *76Hz*. F3 is pressure load of *25KPa* applied at *95Hz*.

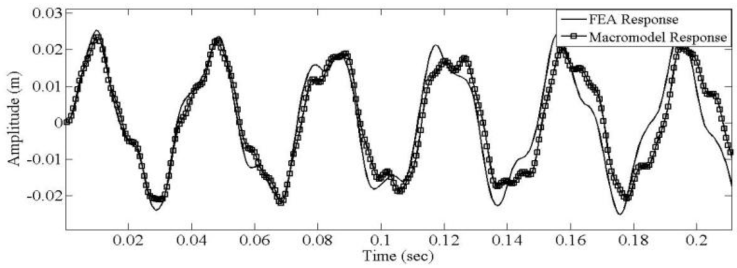
Cases	M [kg]X10^{-3}		K [N/m]		F [N]					
					F1		F2		F3	
	Modes		Modes		Modes		Modes		Modes	
	1	2	1	2	1	2	1	2	1	2
Uncracked	6.88	6.82	190	7448	190.10	219.18	95.05	109.5	3291.6	1707.8
Cracked	6.78	6.56	188	7250	113.64	137.29	56.89	69.6	2379.2	1236.6

3.3. Dynamic Analysis in Chaotic Regime using Macromodel

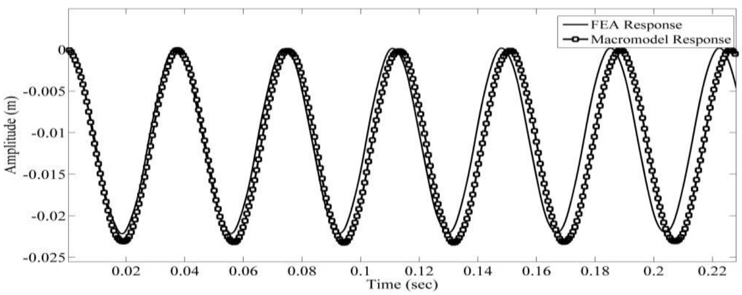
This system is expected to show chaotic behavior due to the presence of discontinuity in the form of breathing crack. Once the macromodel is formulated and robustness is checked under large deformation based nonlinearities, this macromodel is used to predict the chaotic behavior of the system. For identifying chaos, various nonlinear dynamic tools are used such,



(a)



(b)



(c)

Fig. 6. Comparison of FEA and Macromodel Responses of Cracked Beam for a) Different Forcing Frequency (*800 rad/s*) b) Different Loading Amplitude and Frequency (*f = 1000*sin (475*t)*).c) Pressure Load.

As, Poincare sections, bifurcation diagrams etc. Poincare sections are plotted along with bifurcation diagram by using macromodel ODEs' for periodic loading case. Fig. 7 shows the phase portrait and Poincare sections for the cracked cantilever beam subjected to periodic load. The dynamics of the system doesn't seem to settle down to a periodic orbit and the Poincare map obtained using FEA simulation looks like a typical strange attractor for chaotic systems[10].

Table 2. Time and Error Analysis Results for Different Loading Conditions.

Cases	FEA Simulation Time (sec)	Macromodel Simulation Time (sec)	Time Reduction Factor	Error in Responses (%)
Undamped	400	1.86	215	0.036
Damped	421	1.89	225	0.0053
Changed Forcing Frequency	442	1.98	222	0.031
Changed Forcing Amplitude and Frequency	387	1.90	204	0.024
Pressure	309	2.03	153	0.068

Next the Poincare map with varying forcing amplitudes is plotted in order to determine parameter values that lead to the bifurcation of periodic solution leading to chaos. The resulting bifurcation diagram shown in Fig. 8 depicts regions of periodic behavior interspersed with chaotic region. The systems takes a period doubling route to chaos. This diagram when compared against a similarly obtained diagram using the ROM, shows very good qualitative agreement. The quantitative measures of the chaotic system such as the Lyapunov exponent were not calculated and compared at present.

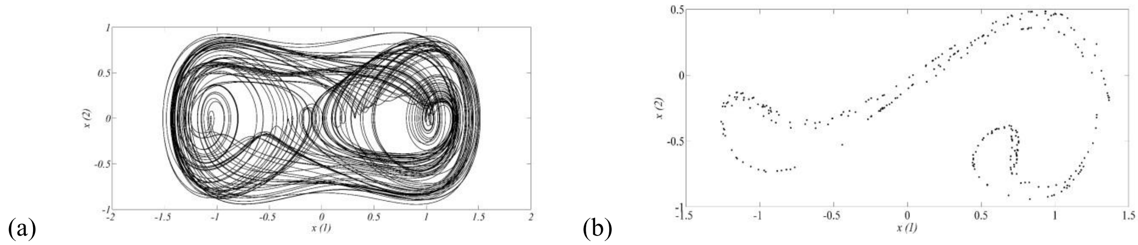


Fig. 7. Poincare map of the response of cracked beam with periodic loading, showing the strange attractor like shape.

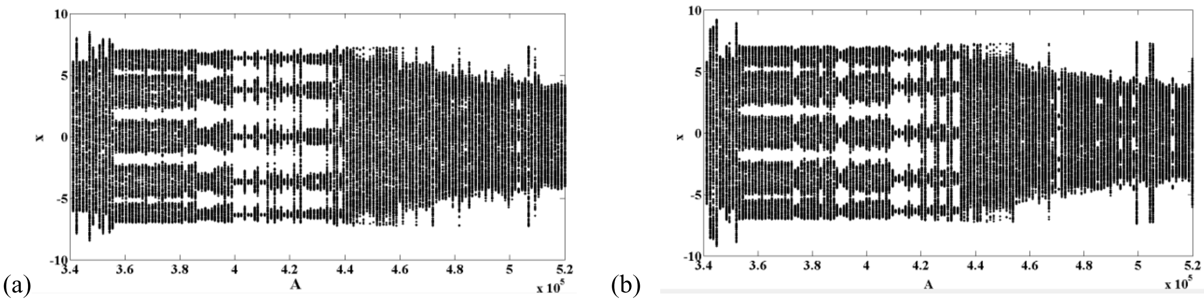


Fig. 8. Bifurcation Diagram for Damped Cracked Beam Case. Obtained from a) FEA b) ROM.

4. Conclusions

A robust and efficient reduced order model for cracked cantilever beam was obtained in order to reduce simulation time and cost. This macromodel is obtained using projection of system dynamics onto the global basis functions called

POD. Results of this macromodel were compared with FEM results. Nonlinearity in the system due to breathing crack and large deformations is effectively captured. The nonlinear dynamic response is obtained using only first two POD modes, thus reducing number of degrees of freedom and hence, simulation time. It is observed that, natural frequency of the cracked beam reduces due to presence of local flexibility in the form of breathing crack and is observed from FFT of the forced vibration response. Time and error analysis shows that average time reduction factor achieved is ~ *200* for breathing crack model with the error being as low as *0.022%.* Further, robustness of this constructed macromodel is checked by applying different loading conditions keeping other modal parameters constant. Chaotic behavior observed in the full scale model using bifurcation diagram with forcing amplitude variation is qualitatively reproduced using the ROM.

References

[1] H. Nahvi, M. Jabbari., Crack detection in beams using experimental modal data and finite element model, International Journal of Mechanical Sciences. 47 (2005) 1477-1497.

[2] Murat Kisa, M. Arif Gurel, Free vibration analysis of uniform and stepped cracked beams with circular cross sections, International Journal of Engineering Sciences. 45 (2007) 364-380.

[3] Akira Saito, Matthew P. Castanier, Christophe Pierre, Olivier Poudou, Efficient Nonlinear Vibration Analysis of the Forced Response of Rotating Cracked Blades, Journal of Computational and Nonlinear Dynamics. 4 (2009) 011005-1.

[4] Akira Saito, Matthew P. Castanier, Christophe Pierre, Effects of a Cracked Blade on Mistuned Turbine Engine Rotor Vibration, Journal of Vibration and Acoustics. 131 (2009) 061006-1.

[5] M. Chati, R. Rand, S. Mukherjee, Modal analysis of a cracked beam, Journal of Sound and Vibration. 207 (2) (1997) 249-270.

[6] M.H.H. Shen, Y. C. Chu, Vibrations of Beams with a Fatigue Crack, Copters & Stiwctwes. 4.5 (1) (1992) 79-93.

[7] Ugo Andreausa, Paolo Casinib, Fabrizio Vestronia, Nonlinear dynamic analysis of a cracked cantilever beam under harmonic excitation, International Journal of Nonlinear Mechanics. 42 (2007) 566-575.

[8] Elmer S. Hung and Stephen D. Senturia, Generating Efficient Dynamical Models for Microelectromechanical Systems from a Few Finite-Element Simulation Runs, IEEE Journal of Microelectromechanical Systems. 8(3) (1999) 280-289.

[9] Anindya Chatterjee, An introduction to the proper orthogonal decomposition, Current Science. 78(7) (2000) 808-817.

[10] Francis C. Moon, Chaotic and Fractal Dynamics: An Introduction for Applied Scientists and Engineers, Wiley-Interscience, 1992.

[11] J. J. Thomsen, Vibrations and Stability: Advanced Theory, Analysis, and Tools, Springer, 2003.